Journal of Engineering Mathematics **38:** 155–172, 2000.

Application of Cosserat-spectrum theory to the weakly compressible Stokes flow past a sphere

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Received 27 January 1998; accepted in revised form 28 July 1999

Abstract. The Cosserat-spectrum theory, which provides an eigenfunction-expansion solution to the Navier equations of linear elasticity, has recently been applied successfully to a number of problems in elasticity, thermoelasticity and viscoelasticity. In this work the theory's extension to fluid mechanics is explored and the example problem of the weakly compressible Stokes flow past a sphere is solved in closed form.

Key words: Cosserat-spectrum, compressible, Stokes flow, sphere.

1. Introduction and background

The Cosserat-spectrum theory in elasticity was introduced by Cosserat and Cosserat [1] and subsequently received rigorous mathematical attention by Mikhlin [2]. It is only recently that applications of the Cosserat-spectrum theory have appeared in literature. Markenscoff and Paukshto [3] developed a new variational principle in thermoelasticity within the framework of the Cosserat-spectrum theory. It states that the thermoelastic energy is maximized or minimized if a thermoelastic solution is contributed from a Cosserat eigenfunction which corresponds to the maximum or minimum eigenvalue. Therefore, the Cosserat-spectrum theory not only provides a method to solve a thermoelastic problem, but it also reveals some physical insight of the solution to the problem. Liu, Markenscoff and Paukshto [4] used the new energy principle to examine several thermoelastic problems. They also showed the fast convergence of the Cosserat eigenfunctions. Although the solution to a thermoelastic problem is written in the form of the summation of the Cosserat eigenfunctions, only the first few terms are needed. It therefore provides a feasible approach to the solution of practical problems. Markenscoff, Liu and Paukshto [5] demonstrated that the viscoelastic equations in the form of Laplace transforms are essentially the same as their counterparts in elasticity and thus the Cosserat-spectrum theory can be extended to the study of problems in viscoelasticity. They also showed that the viscoelastic-material models are naturally related to the Cosserat eigenfunctions.

The similarity between the Navier equations of elasticity and the Navier-Stokes equations of fluid mechanics suggests the possibility that the Cosserat-spectrum theory will have application to the solution of problems in fluid mechanics. Liu and Plotkin [6] took a step in this direction in their study of the incompressible Stokes flow past a sphere with different free-stream profiles. The assumption of incompressibility, however, leads to a solution of a reduced set of governing equations. The most general version of the equations requires a consideration of the fluid's compressibility. In the present paper we extend the Cosserat-spectrum theory to weakly compressible flow.

An outline of the aspects of the Cosserat-spectrum theory will now be presented. The details are available in [7]. The uniqueness theorem in elasticity [8, pp. 9–22] states that in a bounded domain Ω the homogeneous Navier equation $\Delta\mathbf{u} + \omega\nabla\nabla\cdot\mathbf{u} = 0$, with the homogeneous boundary displacement, admits trivial solutions $\mathbf{u} = \mathbf{0}$ when $\omega > -1$, where $\omega = (\lambda+\mu)/\mu = 1/(1-2\nu)$, λ and μ are the Lame constants, ν the Poisson's ratio, $\partial\Omega$ the surface of Ω, $\mathbf{u}$ the displacement vector.

In a series of papers [7, pp. 225–228] the Cosserats showed that the homogeneous Navier equation admits nontrivial solutions when ω takes some special values $\tilde{\omega}$ under homogeneous boundary conditions. The values $\tilde{\omega}$ and the corresponding non-zero solutions $\tilde{\mathbf{u}}$ are now called the Cosserat eigenvalues and Cosserat eigenvectors, respectively.

The Cosserat-spectrum theory was then fully developed by Mikhlin [2] who proved the completeness and orthogonality of the Cosserat eigenfunctions and represented the displacement field $\mathbf{u}$ for an inhomogeneous problem as a summation of the Cosserat eigenfunctions. For the boundary-value problem of displacement in 3D, the eigenvectors are complete and form three orthogonal subspaces, namely, the discrete eigenvectors $\tilde{\mathbf{u}}_n$, the eigenvectors $\tilde{\mathbf{u}}_n^{(-1)}$ corresponding to the eigenvalue of infinite multiplicity $\tilde{\omega} = -1$, and the eigenvectors $\tilde{\mathbf{u}}_n^{(\infty)}$ corresponding to the eigenvalue of infinite multiplicity $\tilde{\omega} = -\infty$. The solution of the inhomogeneous problem

$$\Delta\mathbf{u} + \omega\nabla\nabla\cdot\mathbf{u} = -\frac{\mathbf{F}}{\mu} \quad \text{in } \Omega, \tag{1.1a}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega, \tag{1.1b}$$

admits the Mikhlin representation

$$\mathbf{u} = \frac{1}{\mu}\sum_n \left\{ \frac{(\mathbf{F}, \tilde{\mathbf{u}}_n^{(-1)})}{1+\omega}\tilde{\mathbf{u}}_n^{(-1)} + (\mathbf{F}, \tilde{\mathbf{u}}_n^{(\infty)})\tilde{\mathbf{u}}_n^{(\infty)} + \frac{\tilde{\omega}_n}{\tilde{\omega}_n - \omega}(\mathbf{F}, \tilde{\mathbf{u}}_n)\tilde{\mathbf{u}}_n \right\}, \tag{1.2a}$$

where

$$(\mathbf{F}, \tilde{\mathbf{u}}) \equiv \int \mathbf{F}\cdot\tilde{\mathbf{u}}\,\mathrm{d}V \tag{1.2b}$$

and $\mathrm{d}V$ is the volume element in 3D.

Consider now the possible extension of the Cosserat-spectrum theory to fluid mechanics. The steady Navier–Stokes equations describing barotropic flows of a compressible viscous fluid have the form

$$\rho(\mathbf{u}\cdot\nabla)\mathbf{u} = \mu\Delta\mathbf{u} + (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) - \nabla p, \tag{1.3a}$$

$$\nabla\cdot(\rho\mathbf{u}) = 0, \qquad p = p(\rho), \tag{1.3b,c}$$

where ρ, $\mathbf{u}$, and p are the density, velocity, and pressure, respectively; μ and λ, the constant coefficients of dynamic and shear viscosity, respectively, satisfying $\mu > 0$, $3\lambda + 2\mu \geqslant 0$.

Comparing the momentum equation (1.3a) with the Navier equation (1.1a), we see that, in order to use the Cosserat-spectrum theory, the nonlinear convective term $\rho(\mathbf{u}\cdot\nabla)\mathbf{u}$ must be absent, but the compressibility term $\nabla\nabla\cdot\mathbf{u}$ must be present. In [6] the special case of incompressible flow is studied. In this paper we consider a low-Reynolds-number flow. A system at low Reynolds number ($\mathrm{Re} \ll 1$) is distinguished by the fact that the nonlinear convective term makes a small contribution to the equation of motion, and may thus be neglected. Consequently, Equation (1.3) is simplified as follows

$$\mu\Delta\mathbf{u} + (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) = \nabla p, \tag{1.4a}$$

$$\nabla \cdot (\rho\mathbf{u}) = 0, \qquad \frac{\mathrm{d}p}{\mathrm{d}\rho} = c^2, \tag{1.4b,c}$$

where a particular equation of state, (1.4c), is used and c, the speed of sound, is a constant. This choice of equation of state is made to simplify the following analysis.

Since the momentum equation (1.4a) has the same form as the Navier equation (1.1a), the Cosserat-spectrum theory could be applicable to the system of Equations (1.4). A question naturally arises as to how small the Reynolds number should be for Equation (1.4a) to be justified; in other words, how can one neglect the nonlinear convective term, while retaining the compressibility term? The scaling and perturbation analysis in Section 2 will address this issue. Section 3 applies the Cosserat-spectrum theory to solve for the first-order perturbation terms due to compressibility. We then illustrate the solution technique by studying the weakly compressible Stokes flow over a sphere with a uniform free stream profile in Section 4.

2. Scaling and perturbation analysis

Let us nondimensionalize the Navier–Stokes equations (1.3), using scaling appropriate to a flow at low Reynolds number (Stokes flow). Consider a domain characterized by the length L and a motion characterized by the velocity U. The proper nondimensional position vector $\overline{\mathbf{x}} = \mathbf{x}/L$ and velocity $\overline{\mathbf{u}} = \mathbf{u}/U$ contain these scales. For a small Reynolds number flow, the pressure force must become large to balance the viscous stresses; thus, the appropriate nondimensional pressure is $\overline{p} = p/(\mu U/L)$. The density is nondimensionalized by $\overline{\rho} = \rho/\rho_0$, where ρ_0 is the constant density for incompressible flow. The Navier–Stokes equations (1.3) are now scaled in dimensionless form as follows

$$\mathrm{Re}(\overline{\mathbf{u}} \cdot \overline{\nabla})\overline{\mathbf{u}} = \overline{\Delta}\overline{\mathbf{u}} + \omega\overline{\nabla}(\overline{\nabla} \cdot \overline{\mathbf{u}}) - \overline{\nabla}\overline{p}, \tag{2.1a}$$

$$\overline{\nabla} \cdot (\overline{\rho}\,\overline{\mathbf{u}}) = \overline{\nabla}\overline{\rho} \cdot \overline{\mathbf{u}} + \overline{\rho}\overline{\nabla} \cdot \overline{\mathbf{u}} = 0, \qquad \frac{\mathrm{d}\overline{p}}{\mathrm{d}\overline{\rho}} = \frac{\mathrm{Re}}{M^2}, \tag{2.1b, c}$$

where $\overline{\nabla}$ and $\overline{\Delta}$ are the dimensionless form of ∇ and Δ, respectively; $\mathrm{Re} = LU\rho_0/\mu$ is the Reynolds number, $M = U/c$ the Mach number, and $\omega = (\lambda + \mu)/\mu$. If the Stokes assumption [9, p. 131] is taken into account, then $\omega = \frac{1}{3}$.

As mentioned in Section 1, the nonlinear convective terms $\mathrm{Re}(\overline{\mathbf{u}} \cdot \overline{\nabla})\overline{\mathbf{u}}$ typically are negligible for flows at low Reynolds number ($\mathrm{Re} \ll 1$). Therefore, Equation (2.1a) reduces to the nondimensionalized form of Equation (1.4a)

$$\overline{\Delta}\overline{\mathbf{u}} + \omega\overline{\nabla}(\overline{\nabla} \cdot \overline{\mathbf{u}}) = \overline{\nabla}\overline{p}. \tag{2.2}$$

For a weakly compressible flow, we assume

$$\varepsilon = \frac{M^2}{\mathrm{Re}} \ll 1 \tag{2.3}$$

and perform a general perturbation expansion for $\bar{\mathbf{u}}$, $\bar{p}$ and $\bar{\rho}$ as follows

$$\bar{\mathbf{u}} = \bar{\mathbf{u}}_{\text{inc}} + \varepsilon\bar{\mathbf{u}}_1 + \varepsilon^2\bar{\mathbf{u}}_2 + \cdots, \tag{2.4a}$$

$$\bar{p} = \bar{p}_{\text{inc}} + \varepsilon\bar{p}_1 + \varepsilon^2\bar{p}_2 + \cdots, \qquad \bar{\rho} = 1 + \varepsilon\bar{\rho}_1 + \varepsilon^2\bar{\rho}_2 + \cdots, \tag{2.4b,c}$$

where $\bar{\mathbf{u}}_{\text{inc}}$, $\bar{p}_{\text{inc}}$ and 1 are the dimensionless velocity, pressure and density for the incompressible flow, respectively, $\bar{\mathbf{u}}_i$, $\bar{p}_i$ and $\bar{\rho}_i$ are the ith order perturbation terms of their counterparts.

We rewrite Equation (2.1c) as $\mathrm{d}\bar{\rho} = \varepsilon\,\mathrm{d}\bar{p}$ and a simple integration gives

$$\bar{\rho} = 1 + \varepsilon\bar{p}, \tag{2.5}$$

where the integration constant 1 is the dimensionless density for $\varepsilon = 0$ (incompressible flow). We then substitute Equations (2.4b,c) in Equations (2.5) and equate powers of ε to get

$$\bar{\rho}_1 = \bar{p}_{\text{inc}}, \qquad \bar{\rho}_{i+1} = \bar{p}_i, \quad i = 1, 2, 3, \ldots. \tag{2.6a,b}$$

Equation (2.6) shows that the $(i+1)$th density perturbation term is equal to the ith pressure perturbation term.

With the use of Equations (2.4a,c), Equation (2.1b) becomes

$$\begin{aligned}
&\bar{\nabla}\cdot(\bar{\rho}\,\bar{\mathbf{u}}) \\
&\quad = (\varepsilon\bar{\nabla}\bar{\rho}_1 + \varepsilon^2\bar{\nabla}\bar{\rho}_2 + \cdots)\cdot(\bar{\mathbf{u}}_{\text{inc}} + \varepsilon\bar{\mathbf{u}}_1 + \cdots) \\
&\qquad + (1 + \varepsilon\bar{\rho}_1 + \cdots)(\bar{\nabla}\cdot\bar{\mathbf{u}}_{\text{inc}} + \varepsilon\bar{\nabla}\cdot\bar{\mathbf{u}}_1 + \cdots) \\
&\quad = \bar{\nabla}\cdot\bar{\mathbf{u}}_{\text{inc}} + \varepsilon(\bar{\nabla}\bar{\rho}_1\cdot\bar{\mathbf{u}}_{\text{inc}} + \bar{\rho}_1\bar{\nabla}\cdot\bar{\mathbf{u}}_{\text{inc}} + \bar{\nabla}\cdot\bar{\mathbf{u}}_1) + O(\varepsilon^2) + \cdots \\
&\quad = 0.
\end{aligned} \tag{2.7}$$

We now equate powers of ε to obtain the leading-order and the first-order perturbation terms of the continuity equation as follows

$$\bar{\nabla}\cdot\bar{\mathbf{u}}_{\text{inc}} = 0, \qquad \bar{\nabla}\bar{\rho}_1\cdot\bar{\mathbf{u}}_{\text{inc}} + \bar{\nabla}\cdot\bar{\mathbf{u}}_1 = 0. \tag{2.8a,b}$$

With the use of Equations (2.4a,b), Equation (2.2) becomes

$$\begin{aligned}
&[\bar{\Delta}\bar{\mathbf{u}}_{\text{inc}} + \omega\bar{\nabla}(\bar{\nabla}\cdot\bar{\mathbf{u}}_{\text{inc}}) - \bar{\nabla}\bar{p}_{\text{inc}}] \\
&\quad + \varepsilon[\bar{\Delta}\bar{\mathbf{u}}_1 + \omega\bar{\nabla}(\bar{\nabla}\cdot\bar{\mathbf{u}}_1) - \bar{\nabla}\bar{p}_1] + O(\varepsilon^2) + \cdots = 0.
\end{aligned} \tag{2.9}$$

Let us now equate powers of ε and use Equation (2.8a) to get the leading-order and the first-order perturbation terms of the momentum equation as follows

$$\bar{\Delta}\bar{\mathbf{u}}_{\text{inc}} = \bar{\nabla}\bar{p}_{\text{inc}}, \qquad \bar{\Delta}\bar{\mathbf{u}}_1 + \omega\bar{\nabla}(\bar{\nabla}\cdot\bar{\mathbf{u}}_1) = \bar{\nabla}\bar{p}_1. \tag{2.10a,b}$$

This approach shows that the perturbation terms appear in the order of $\varepsilon^i = (M^2/\text{Re})^i$, $i = 1, 2, 3\ldots$. The present paper will retain the first perturbation term. Recall that we neglect the nonlinear convective term for Re$\ll 1$. Therefore, the Stokes scaling and perturbation analysis implies that Equation (1.4a) is justified under the limit

$$M^2 \ll \text{Re} \ll 1. \tag{2.11}$$

Equation (2.11) gives rigorous requirements for a weakly compressible flow. The potential application may be found in a few examples such as a polytropic gas [10] and slider air-bearings [11], where the compressibility effects were considered for flows at small Reynolds number. It is of theoretical interest to explore the extension of the Cosserat-spectrum theory to fluid mechanics. The mathematical aspects of compressible flows at low Reynolds number have been studied by Kazhikhov [12] and Weigant and Kazhikhov [13]. Kozhevnikov [14] studied the connection between the Stokes problems in hydrodynamics and boundary-value problems in elasticity.

3. Application of the Cosserat-spectrum theory to compressible Stokes flow

The present paper extends the Cosserat-spectrum theory to study the first-order perturbation terms of compressible Stokes flow, which are described by Equation (2.2) and Equations (2.1b,c). We now rewrite these equations in dimensional form for a domain Ω as

$$\Delta \mathbf{u} + \omega \nabla\nabla \cdot \mathbf{u} = \frac{\nabla p}{\mu} \quad \text{in } \Omega, \tag{3.1a}$$

$$\nabla \cdot (\rho \mathbf{u}) = 0 \quad \text{in } \Omega, \qquad \mathrm{d}p/\mathrm{d}\rho = c^2 \quad \text{in } \Omega. \tag{3.1b,c}$$

The velocity is prescribed on the boundary $\partial\Omega$ as

$$\mathbf{u} = \mathbf{u}_b \quad \text{on } \partial\Omega. \tag{3.2}$$

For a weakly compressible flow, we approximate the solutions to the system of Equation (3.1) to be the sum of the corresponding incompressible Stokes solutions and their first-order perturbation terms, namely

$$\mathbf{u} \approx \mathbf{u}_{\text{inc}} + \varepsilon \mathbf{u}_1, \qquad p \approx p_{\text{inc}} + \varepsilon p_1, \qquad \rho \approx \rho_0 + \varepsilon \rho_1, \tag{3.3a,b,c}$$

where $\mathbf{u}_{\text{inc}}$, p_{inc}, and ρ_0 are the solutions to the incompressible counterparts; $\mathbf{u}_1$, p_1, and ρ_1 are their first-order perturbation terms, respectively, and $\varepsilon = M^2/\text{Re}$.

As analyzed in the previous section, the leading-order terms are associated with the solutions to the imcompressible counterparts $\mathbf{u}_{\text{inc}}$, p_{inc}, and ρ_0 as follows

$$\Delta \mathbf{u}_{\text{inc}} = \frac{\nabla p_{\text{inc}}}{\mu} \quad \text{in } \Omega, \tag{3.4a}$$

$$\nabla \cdot \mathbf{u}_{\text{inc}} = 0 \quad \text{in } \Omega, \qquad \rho = \rho_0 \quad \text{in } \Omega, \tag{3.4b,c}$$

with the boundary condition

$$\mathbf{u}_{\text{inc}} = \mathbf{u}_b \quad \text{on } \partial\Omega. \tag{3.4d}$$

The first-order perturbation terms $\mathbf{u}_1$, p_1, and ρ_1 are the dimensional form of Equation (2.10b), Equation (2.8b) and Equation (2.6a), namely

$$\Delta \mathbf{u}_1 + \omega \nabla(\nabla \cdot \mathbf{u}_1) = \frac{\nabla p_1}{\mu} \quad \text{in } \Omega, \tag{3.5a}$$

$$\nabla\rho_1 \cdot \mathbf{u}_{\text{inc}} + \rho_0 \nabla \cdot \mathbf{u}_1 = 0 \quad \text{in } \Omega, \qquad \rho_1 = \frac{\rho_0 L}{\mu U} p_{\text{inc}} \quad \text{in } \Omega, \tag{3.5b, c}$$

with the boundary condition

$$\mathbf{u}_1 = 0 \quad \text{on } \partial\Omega. \tag{3.5d}$$

We shall now use Equations (3.5b,c) to write the linear continuity equation for the perturbation term $\mathbf{u}_1$ as

$$\nabla \cdot \mathbf{u}_1 = -\frac{L}{\mu U} \nabla p_{\text{inc}} \cdot \mathbf{u}_{\text{inc}}. \tag{3.6}$$

In the Cosserat-spectrum theory, Equation (3.5a) is the Cosserat-eigenvalue problem of the first kind. Using the equivalent body force $\mathbf{F} = -\nabla p_1$ in the Mikhlin-representation equation (1.2), we have

$$\mathbf{u}_1 = -\frac{1}{\mu} \sum_n \left\{ \frac{(\nabla p_1, \tilde{\mathbf{u}}_n^{(-1)})}{\omega + 1} \tilde{\mathbf{u}}_n^{(-1)} + (\nabla p_1, \tilde{\mathbf{u}}_n^{(\infty)}) \tilde{\mathbf{u}}_n^{(\infty)} + \frac{\tilde{\omega}_n (\nabla p_1, \tilde{\mathbf{u}}_n)}{(\tilde{\omega}_n - \omega)} \tilde{\mathbf{u}}_n \right\}, \tag{3.7a}$$

$$(\nabla p_1, \tilde{\mathbf{u}}) \equiv \int \nabla p_1 \cdot \tilde{\mathbf{u}} \, \mathrm{d}V. \tag{3.7b}$$

By taking $(\nabla p_1, \tilde{\mathbf{u}}_n^{(\infty)}) = 0$ into account [6], we may reduce Equation (3.7) to

$$\mathbf{u}_1 = \sum_n f_n \tilde{\mathbf{u}}_n + \sum_n \sum_m f_{nm}^{(-1)} \tilde{\mathbf{u}}_{nm}^{(-1)}, \tag{3.8a}$$

where

$$f_n \equiv \frac{\tilde{\omega}_n (\nabla p_1, \tilde{\mathbf{u}}_n)}{\mu(\omega - \tilde{\omega}_n)}, \qquad f_{nm}^{(-1)} \equiv -\frac{(\nabla p_1, \tilde{\mathbf{u}}_{nm}^{(-1)})}{\mu(\omega + 1)}, \tag{3.8b,c}$$

are constants to be determined. In general, there exist infinite subspaces of the Cosserat eigenvectors $\tilde{\mathbf{u}}_{nm}^{(-1)}$ associated with $\tilde{\omega} = -1$. A new index parameter m is introduced in Equation (3.8) in order to specify that $\tilde{\mathbf{u}}_{nm}^{(-1)}$ is the (n, m) component in the infinite orthogonal subspaces of $\tilde{\mathbf{u}}^{(-1)}$ [7, pp. 189–207].

Substituting Equation (3.8a) in Equation (3.5a) yields

$$\sum_n f_n [\Delta \tilde{\mathbf{u}}_n + \omega \nabla\nabla \cdot \tilde{\mathbf{u}}_n] + \sum_n \sum_m f_{nm}^{(-1)} [\Delta \tilde{\mathbf{u}}_{nm}^{(-1)} + \omega \nabla\nabla \cdot \tilde{\mathbf{u}}_{nm}^{(-1)}] = \frac{\nabla p_1}{\mu}. \tag{3.9}$$

We now recall that the Cosserat eigenvalues $\tilde{\omega}_n$ or $\tilde{\omega} = -1$ and their corresponding eigenvectors $\tilde{\mathbf{u}}_n$ or $\tilde{\mathbf{u}}_{nm}^{(-1)}$ satisfy the homogeneous Navier equations

$$\nabla \tilde{\mathbf{u}}_n + \tilde{\omega}_n \nabla\nabla \cdot \tilde{\mathbf{u}}_n = 0, \qquad \nabla \tilde{\mathbf{u}}_{nm}^{(-1)} - \nabla\nabla \cdot \tilde{\mathbf{u}}_{nm}^{(-1)} = 0. \tag{3.10a,b}$$

The summation of Equation (3.10) over index n and m gives

$$\sum_n f_n [\Delta \tilde{\mathbf{u}}_n + \tilde{\omega}_n \nabla\nabla \cdot \tilde{\mathbf{u}}_n] = 0, \tag{3.11a}$$

$$\sum_n \sum_m f_{nm}^{(-1)}[\Delta \tilde{\mathbf{u}}_{nm}^{(-1)} - \nabla\nabla \cdot \tilde{\mathbf{u}}_{nm}^{(-1)}] = 0. \tag{3.11b}$$

Subtraction of Equation (3.9) from Equation (3.11) gives

$$\sum_n f_n(\omega - \tilde{\omega}_n)\nabla\nabla \cdot \tilde{\mathbf{u}}_n + \sum_n \sum_m f_{nm}^{(-1)}(\omega + 1)\nabla\nabla \cdot \tilde{\mathbf{u}}_{nm}^{(-1)} = \frac{\nabla p_1}{\mu}. \tag{3.12}$$

Integration of Equation (3.12) gives the pressure perturbation term

$$p_1 = \mu \sum_n f_n(\omega - \tilde{\omega}_n)\nabla \cdot \tilde{\mathbf{u}}_n + \mu \sum_n \sum_m f_{nm}^{(-1)}(\omega + 1)\nabla \cdot \tilde{\mathbf{u}}_{nm}^{(-1)}. \tag{3.13}$$

The linear continuity equation (3.6) is now employed to evaluate the coefficients f_n and $f_{nm}^{(-1)}$. With the use of Equation (3.8a), Equation (3.6) becomes

$$\sum_n f_n \nabla \cdot \tilde{\mathbf{u}}_n + \sum_n \sum_m f_{nm}^{(-1)} \nabla \cdot \tilde{\mathbf{u}}_{nm}^{(-1)} = -\frac{L}{\mu U}\nabla p_{\text{inc}} \cdot \mathbf{u}_{\text{inc}}. \tag{3.14}$$

Each term in Equation (3.14) is multiplied by $\nabla \cdot \tilde{\mathbf{u}}_k$ or $\nabla \cdot \tilde{\mathbf{u}}_{kl}^{(-1)}$, then integrated over the domain Ω, and with the use of the orthogonality conditions [2] for the first boundary-value problem, we have

$$f_n = \frac{L\tilde{\omega}_n}{\mu U} \int \nabla p_{\text{inc}} \cdot \mathbf{u}_{\text{inc}} \nabla \cdot \tilde{\mathbf{u}}_n \, \mathrm{d}V, \tag{3.15a}$$

$$f_{nm}^{(-1)} = -\frac{L}{\mu U} \int \nabla p_{\text{inc}} \cdot \mathbf{u}_{\text{inc}} \nabla \cdot \tilde{\mathbf{u}}_{nm}^{(-1)} \, \mathrm{d}V. \tag{3.15b}$$

The pressure perturbation term p_1 represented by Equation (3.13) can be further simplified. From Equation (3.14) we have

$$\sum_n \sum_m f_{nm}^{(-1)} \nabla \cdot \tilde{\mathbf{u}}_{nm}^{(-1)} = -\frac{L}{\mu U}\nabla p_{\text{inc}} \cdot \mathbf{u}_{\text{inc}} - \sum_n f_n \nabla \cdot \tilde{\mathbf{u}}_n. \tag{3.16}$$

With the use of Equation (3.16), Equation (3.13) becomes

$$p_1 = -\mu \sum_n f_n(1 + \tilde{\omega}_n)\nabla \cdot \tilde{\mathbf{u}}_n - \frac{(\omega + 1)L}{U}\nabla p_{\text{inc}} \cdot \mathbf{u}_{\text{inc}}. \tag{3.17}$$

To solve this compressible-flow problem, one needs to add the perturbation terms $\varepsilon \mathbf{u}_1$, εp_1, and $\varepsilon \rho_1$ to the incompressible counterparts $\mathbf{u}_{\text{inc}}$, p_{inc}, and ρ_0. The perturbation terms are associated with Equation (3.8a), Equation (3.17) and Equation (3.5c), respectively.

It is important to mention that $f_{nm}^{(-1)} \neq 0$ for compressible flow in general. Although it has no contribution in incompressible flow, the infinite orthogonal subspaces of the Cosserat eigenfunctions $\tilde{\mathbf{u}}^{(-1)}$ associated with eigenvalue $\tilde{\omega} = -1$ play an important role in compressible flow. In other words, the incompressible pressure p_{inc} is harmonic, but the compressible flow pressure $p = p_{\text{inc}} + \varepsilon p_1$ is nonharmonic.

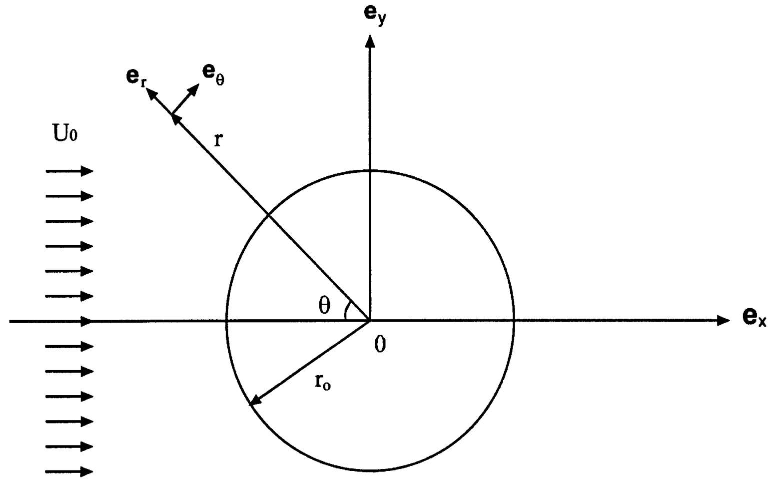


Figure 1. Spherical coordinate system for flow over a sphere.

4. Example: Uniform flow past a sphere

To illustrate the solution technique described above, we now study the compressible Stokes flow past a sphere with a uniform free-stream profile. The Cosserat eigenvalues and eigenvectors for the first boundary-value problem of a spherical rigid inclusion are presented in the Appendix.

The spherical coordinate system is shown in Figure 1. Note that the angle φ is not shown. For this specific flow, the characteristic velocity $U = U_0$ and characteristic length $L = r_0$, where U_0 is the free-stream velocity and r_0 the radius of the sphere. Consequently, the Reynolds number $\mathrm{Re} = r_0 U_0 \rho_0/\mu$, the Mach number $M = U_0/c$, and the perturbation parameter $\varepsilon = M^2/\mathrm{Re} = \mu U_0/r_0\rho_0 c^2$.

As described in the previous section, the solutions of a compressible flow are approximated as the sum of its incompressible counterparts and the corresponding perturbation terms. We now study the perturbation terms for the compressible flow over a sphere with uniform free-stream profile. Referring to the pressure in the incompressible flow [9, p. 688] we calculate its gradient as follows

$$p_{\mathrm{inc}} = p_0 + \frac{3\mu r_0 U_0}{2r^2}\cos\theta, \tag{4.1}$$

$$\nabla p_{\mathrm{inc}} = -\frac{3\mu r_0 U_0}{r^3}(\cos\theta\,\mathbf{e}_r + \tfrac{1}{2}\sin\theta\,\mathbf{e}_\theta). \tag{4.2}$$

The velocity field of the incompressible flow [9, p. 688] can be expressed as follows

$$\mathbf{u}_{\mathrm{inc}} = U_0\left(-1 + \frac{3r_0}{2r} - \frac{r_0^3}{2r^3}\right)\cos\theta\,\mathbf{e}_r + U_0\left(1 - \frac{3r_0}{4r} - \frac{r_0^3}{4r^3}\right)\sin\theta\,\mathbf{e}_\theta. \tag{4.3}$$

The scalar product of $\nabla p_{\mathrm{inc}} \cdot \mathbf{u}_{\mathrm{inc}}$ is now written as

$$\nabla p_{\mathrm{inc}} \cdot \mathbf{u}_{\mathrm{inc}} = -\frac{3\mu U_0^2}{r_0^2}[\tfrac{1}{4}(s^4 - s^6) - (s^3 - \tfrac{5}{4}s^4 + \tfrac{1}{4}s^6)P_2(\cos\theta)], \tag{4.4}$$

where $s = r_0/r$ and $P_2(\cos\theta)$ is the Legendre polynomial of degree 2.

We now use Equation (4.4) and Equation (A3) in conjunction with Equation (3.15a) to evaluate the coefficients f_n as follows

$$f_2 = \frac{3\pi C_2 r_0^2 U_0}{2} \quad \text{and} \quad C_2 = \pm\sqrt{\frac{1}{16\pi r_0^3}}, \tag{4.5a}$$

$$f_n = 0, \quad n = 1, 3, 4, \ldots. \tag{4.5b}$$

Inserting Equation (4.4) into Equation (3.15b) to obtain the coefficients $f_{nm}^{(-1)}$, we have

$$f_{nm}^{(-1)} = \frac{3U_0}{r_0}\int [\tfrac{1}{4}(s^4 - s^6) - (s^3 - \tfrac{5}{4}s^4 + \tfrac{1}{4}s^6)P_2]\nabla\cdot\tilde{\mathbf{u}}_{nm}^{(-1)}\,\mathrm{d}V, \tag{4.6}$$

where

$$\mathrm{d}V = \frac{2\pi r_0^3 \sin\theta}{s^4}\,\mathrm{d}s\,\mathrm{d}\theta, \quad s\in[0,1], \quad \theta\in[0,\pi],$$

is the volume element in the spherical coordinate system (s,θ) [7, pp. 189–207].

Next, we use $\nabla\cdot\tilde{\mathbf{u}}_{nm}^{(-1)} \propto P_n(\cos\theta)$ and the orthogonal property of the Legendre polynomials to get

$$f_{0m}^{(-1)} = \frac{3U_0}{4r_0}\int (s^4 - s^6)\nabla\cdot\tilde{\mathbf{u}}_{0m}^{(-1)}\,\mathrm{d}V, \tag{4.7a}$$

$$f_{2m}^{(-1)} = -\frac{3U_0}{r_0}\int (s^3 - \tfrac{5}{4}s^4 + \tfrac{1}{4}s^6)P_2\nabla\cdot\tilde{\mathbf{u}}_{2m}^{(-1)}\,\mathrm{d}V, \tag{4.7b}$$

$$f_{nm}^{(-1)} = 0, \quad n \neq 0, 2. \tag{4.7c}$$

We now insert Equation (A7) into Equation (4.7a) to evaluate the coefficient $f_{0m}^{(-1)}$ and get

$$f_{0m}^{(-1)} = 3\pi C_{0m} r_0^2 U_0 \int_{s=0}^{1} (s^2 - s^4) J_{0m}(s)\,\mathrm{d}s. \tag{4.8}$$

Since the Jacobi polynomials are complete, we can write

$$s^2 - s^4 = \tfrac{2}{15}J_{00} + \tfrac{1}{5}J_{01} - \tfrac{5}{7}J_{02} - 2J_{03} - J_{04}. \tag{4.9}$$

The orthogonal property of the Jacobi polynomials $J_{0m}(s)$ is used to get

$$f_{00}^{(-1)} = \frac{2\pi C_{00} h_{00} r_0^2 U_0}{5}, \qquad f_{01}^{(-1)} = \frac{3\pi C_{01} h_{01} r_0^2 U_0}{5}, \tag{4.10a,b}$$

$$f_{02}^{(-1)} = -\frac{15\pi C_{02} h_{02} r_0^2 U_0}{7}, \qquad f_{03}^{(-1)} = -6\pi C_{03} h_{03} r_0^2 U_0, \tag{4.10c,d}$$

$$f_{04}^{(-1)} = -3\pi C_{04} h_{04} r_0^2 U_0, \qquad f_{0m}^{(-1)} = 0 \quad (m \geqslant 5). \tag{4.10e,f}$$

Inserting Equation (A12) into (4.7b) to evaluate the coefficient $f_{2m}^{(-1)}$, we have

$$f_{2m}^{(-1)} = -\frac{12\pi C_{2m} r_0^2 U_0}{5} \int_{s=0}^{1} (1 - \tfrac{5}{4}s + \tfrac{1}{4}s^3) J_{2m}(s) s^2 \, \mathrm{d}s. \tag{4.11}$$

The completeness of the Jacobi polynomials is used to evaluate

$$1 - \tfrac{5}{4}s + \tfrac{1}{4}s^3 = \tfrac{3}{16}J_{20} - \tfrac{25}{28}J_{21} + \tfrac{15}{32}J_{22} + \tfrac{1}{4}J_{23}. \tag{4.12}$$

Use of the orthogonal property of the Jacobi polynomials $J_{2m}(s)$ yields

$$f_{21}^{(-1)} = \frac{15\pi C_{21} h_{21} r_0^2 U_0}{7}, \qquad f_{22}^{(-1)} = -\frac{9\pi C_{22} h_{22} r_0^2 U_0}{8}, \tag{4.13a,b}$$

$$f_{23}^{(-1)} = -\frac{3\pi C_{23} h_{23} r_0^2 U_0}{5}, \qquad f_{2m}^{(-1)} = 0 \quad (m \geqslant 4). \tag{4.13c,d}$$

The velocity perturbation term due to the contribution from the discrete Cosserat eigenfunctions $\tilde{\mathbf{u}}_n$ is as follows

$$f_2 \tilde{\mathbf{u}}_2 = \frac{3U_0}{32}(s^2 - s^4)\left(-3P_2\mathbf{e}_r + \frac{\mathrm{d}P_2}{\mathrm{d}\theta}\mathbf{e}_\theta\right). \tag{4.14}$$

With the use of Equation (4.10) and $\tilde{\mathbf{u}}_{0m}^{(-1)}$, defined by Equations (A10a,c,e,g,i), we evaluate the second part of Equation (3.8a) to obtain the velocity perturbations term due to the contribution from the Cosserat eigenfunctions $\tilde{\mathbf{u}}_{0m}^{(-1)}$ as follows

$$\begin{aligned} &f_{00}^{(-1)}\tilde{\mathbf{u}}_{00}^{(-1)} + f_{01}^{(-1)}\tilde{\mathbf{u}}_{01}^{(-1)} + f_{02}^{(-1)}\tilde{\mathbf{u}}_{02}^{(-1)} + f_{03}^{(-1)}\tilde{\mathbf{u}}_{03}^{(-1)} + f_{04}^{(-1)}\tilde{\mathbf{u}}_{04}^{(-1)} \\ &\quad = -U_0\left(\frac{s}{14} - \frac{137s^2}{70} + \frac{57s^3}{28} + \frac{3s^4}{5} - \frac{3s^5}{4} - \frac{32s^2\log(s)}{35}\right)\mathbf{e}_r. \end{aligned} \tag{4.15a}$$

Similarly, by using Equation (4.13) and $\tilde{\mathbf{u}}_{2m}^{(-1)}$, defined by Equations (A15a,c,e), we obtain the velocity perturbations term due to the contribution from the Cosserat eigenfunctions $\tilde{\mathbf{u}}_{2m}^{(-1)}$ as follows

$$f_{21}^{(-1)}\tilde{\mathbf{u}}_{21}^{(-1)} + f_{22}^{(-1)}\tilde{\mathbf{u}}_{22}^{(-1)} + f_{23}^{(-1)}\tilde{\mathbf{u}}_{23}^{(-1)} =$$

$$= -\frac{5U_0}{2}\left(\frac{37s^2}{392} - \frac{155s^3}{392} - \frac{81s^4}{196} + \frac{5s^5}{7} - \frac{9s^4\log(s)}{16}\right) P_2\mathbf{e}_r$$

$$+\frac{5U_0}{4}\left(\frac{37s^2}{196} - \frac{155s^3}{392} - \frac{59s^4}{392} + \frac{5s^5}{14} - \frac{3s^4\log(s)}{8}\right)\frac{\mathrm{d}P_2}{\mathrm{d}\theta}\mathbf{e}_\theta. \tag{4.15b}$$

The velocity perturbation term is therefore expressed in closed form as follows

$$\begin{aligned}\varepsilon\mathbf{u}_1 \;=\; & \frac{3\varepsilon U_0}{32}(s^2 - s^4)\left(-3P_2\mathbf{e}_r + \frac{\mathrm{d}P_2}{\mathrm{d}\theta}\mathbf{e}_\theta\right)\\ & -\varepsilon U_0\left(\frac{s}{14} - \frac{137s^2}{70} + \frac{57s^3}{28} + \frac{3s^4}{5} - \frac{3s^5}{4} - \frac{32s^2\log(s)}{35}\right)\mathbf{e}_r\\ & -\frac{5\varepsilon U_0}{2}\left(\frac{37s^2}{392} - \frac{155s^3}{392} - \frac{81s^4}{196} + \frac{5s^5}{7} - \frac{9s^4\log(s)}{16}\right)P_2\mathbf{e}_r\\ & +\frac{5\varepsilon U_0}{4}\left(\frac{37s^2}{196} - \frac{155s^3}{392} - \frac{59s^4}{392} + \frac{5s^5}{14} - \frac{3s^4\log(s)}{8}\right)\frac{\mathrm{d}P_2}{\mathrm{d}\theta}\mathbf{e}_\theta.\end{aligned} \tag{4.16}$$

By using Equation (3.17) to calculate the pressure perturbation term, we get

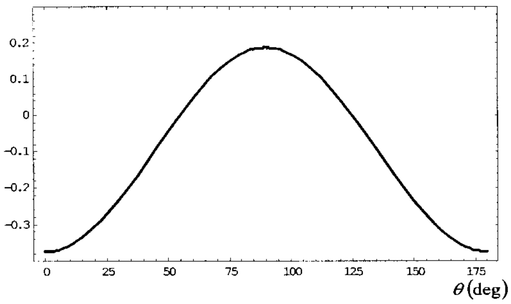
$$\varepsilon p_1 = \frac{3\varepsilon(\omega+1)\mu U_0}{4r_0}[(s^4 - s^6) - (4s^3 - 5s^4 + s^6)P_2] - \frac{3\varepsilon\mu U_0}{8r_0}s^3 P_2. \tag{4.17}$$

Now we use Equation (3.5c) to write the density perturbation term as

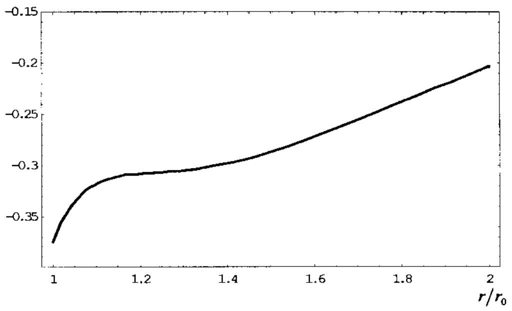
$$\varepsilon\rho_1 = \frac{3\varepsilon\rho_0}{2}s^2\cos\theta. \tag{4.18}$$

Results for the pressure and velocity perturbation terms are shown in the following figures. In Figure 2 the nondimensional pressure $\overline{p}_1 = p_1/(\mu U_0/r_0)$ has been plotted for $\omega = \frac{1}{3}$. Figure 2(a) shows $\overline{p}_1$ on the surface of the sphere. Since the compressible pressure perturbation is proportional to $P_2(\cos\theta)$ on the surface, we see that $\overline{p}_1$ is symmetric about $\theta = 90°$. Figures 2(b,c) show the pressures *vs.* the nondimensional radial distance r/r_0 at $\theta = 0^0$ (or $180°$), $90°$, respectively. At $\theta = 90°$, while the incompressible pressure is identically zero, the compressible pressure is not. In Figure 3 the nondimensional velocity perturbation components $\overline{u}_{1r} = u_{1r}/U_0$ and $\overline{u}_{1\theta} = u_{1\theta}/U_0$ have been plotted *vs.* r/r_0 at different values of θ. Figures 3(a–c) show the radial velocity components $\overline{u}_{1r}$ at $\theta = 0°, 90°, 180°$, respectively. Figures 3(d–e) show the tangential velocity components $\overline{u}_{1\theta}$ at $\theta = 45°, 135°$, respectively. The tangential velocity component $\overline{u}_{1\theta}$ is zero at $\theta = 0°$, $\theta = 90°$ and $\theta = 180°$. At $\theta = 90°$, while the radial velocity component is identically zero for incompressible flow, it is not zero for the weakly compressible flow.

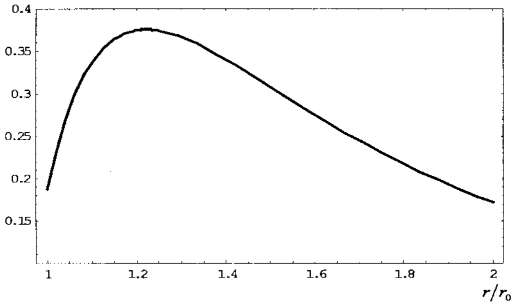
It should be stressed that the only dependence on ω comes from the pressure perturbation term, while the velocity and density perturbation terms are independent of it.



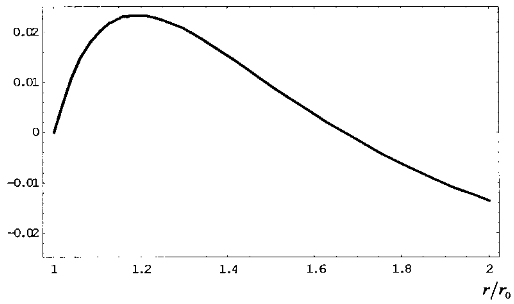
(2a) Dimensionless pressure $\overline{p}_1$ around the surface ($r = r_0$).



(2b) Dimensionless pressure $\overline{p}_1$ *vs.* r/r_0 ($\theta = 0°$, $180°$).

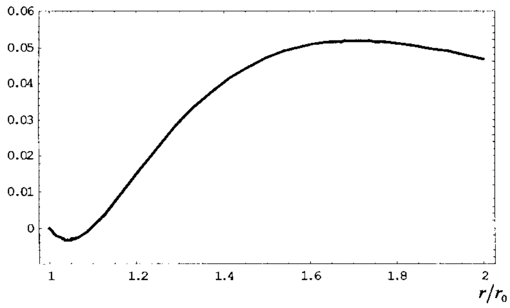


(2c) Dimensionless pressure $\overline{p}_1$ *vs.* r/r_0 ($\theta = 90°$).

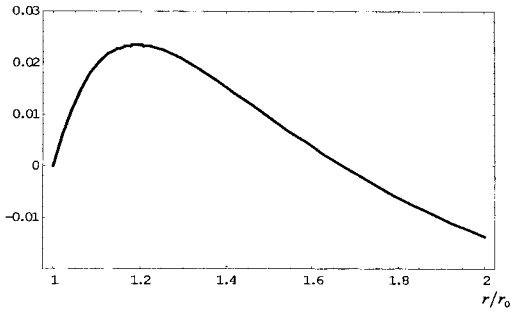


(3a) Dimensionless velocity $\overline{u}_{1r}$ *vs.* r/r_0 ($\theta = 0°$).

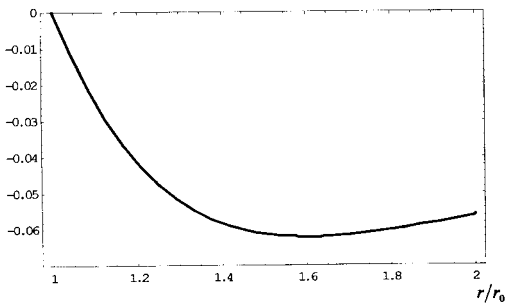
Figure 2. Dimensionless pressure perturbation $\overline{p}_1$.



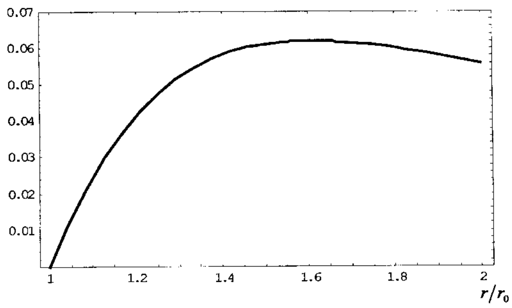
(3b) Dimensionless velocity $\overline{u}_1$ *vs.* r/r_0 ($\theta = 90°$).



(3c) Dimensionless velocity $\overline{u}_{1r}$ *vs.* r/r_0 ($\theta = 180°$).



(3d) Dimensionless velocity $\overline{u}_{1\theta}$ *vs.* r/r_0 ($\theta = 45°$).



(3e) Dimensionless velocity $\overline{u}_{1\theta}$ *vs.* r/r_0 ($\theta = 135°$).

Figure 3. Dimensionless velocity perturbation $\overline{u}_{1r}$ and $\overline{u}_{1\theta}$ *vs.* r/r_0.

Now let us compute the drag force for the weakly compressible flow. The stress $\boldsymbol{\sigma}$ applied to the sphere by the fluid is given by

$$\boldsymbol{\sigma} = p\mathbf{I} - (2\mu\boldsymbol{\varepsilon} + \lambda \text{ div } \boldsymbol{\varepsilon}\mathbf{I}), \tag{4.19}$$

where $\boldsymbol{\varepsilon}$ is the strain rate tensor, and $\mathbf{I}$ is the identity tensor. The traction acting on the surface of the sphere is given by

$$\mathbf{t} = \boldsymbol{\sigma} \cdot \mathbf{n} = [p - (2\mu\varepsilon_{rr} + \lambda\varepsilon_{kk})]\mathbf{e}_r - 2\mu\varepsilon_{r\theta}\mathbf{e}_\theta. \tag{4.20}$$

The traction component in the x-direction will be

$$t_x = \mathbf{t} \cdot \mathbf{e}_x = -[p - (2\mu\varepsilon_{rr} + \lambda\varepsilon_{kk}] \cos\theta - 2\mu\varepsilon_{r\theta} \sin\theta \tag{4.21}$$

and the drag force is therefore given by

$$F_D = \int t_x|_{r=r_0} \, \mathrm{d}s, \tag{4.22}$$

where $\mathrm{d}s = 2\pi r_0^2 \sin\theta \, \mathrm{d}\theta, \theta \in [0, \pi]$.

The pressure and velocity for the weakly compressible flow are recalled as follows

$$p = p_{\text{inc}} + \varepsilon p_1, \tag{4.23}$$

$$\mathbf{u} = \mathbf{u}_{\text{inc}} + \varepsilon f_2 \tilde{\mathbf{u}}_2 + \varepsilon \sum_m f_{0m}^{(-1)} \tilde{\mathbf{u}}_{0m}^{(-1)} + \varepsilon \sum_m f_{2m}^{(-1)} \tilde{\mathbf{u}}_{2m}^{(-1)}. \tag{4.24}$$

The drag force contributed by the incompressible components p_{inc} and $\mathbf{u}_{\text{inc}}$ is given by [9 p. 689]

$$F_{D\text{inc}} = 6\pi\mu U_0 r_0. \tag{4.25}$$

The drag force contributed from the pressure perturbation term $p_1 \propto P_2(\cos\theta)$ is proportional to

$$\int_0^\pi P_2(\cos\theta) \cos\theta \sin\theta \, \mathrm{d}\theta = 0. \tag{4.26}$$

We now evaluate the drag force contributed from the velocity perturbation term $f_2\tilde{\mathbf{u}}_2 \propto f(r)P_2\mathbf{e}_r + g(r)(\mathrm{d}P_2/\mathrm{d}\theta)\mathbf{e}_\theta$, where the functions $f(r)$ and $g(r)$ vanish on $r = r_0$. The strain-rate components on the surface are therefore proportional to

$$\varepsilon_{rr}|_{r=r_0} \propto P_2(\cos\theta), \qquad \varepsilon_{\theta\theta}|_{r=r_0} = 0, \tag{4.27a,b}$$

$$\varepsilon_{\phi\phi}|_{r=r_0} = 0, \qquad \varepsilon_{r\theta}|_{r=r_0} \propto \frac{\mathrm{d}P_2(\cos\theta)}{\mathrm{d}\theta}, \tag{4.27c,d}$$

and the stress components are thus proportional to

$$\sigma_{rr}|_{r=r_0} \propto P_2(\cos\theta), \qquad \sigma_{r\theta}|_{r=r_0} \propto \frac{\mathrm{d}P_2(\cos\theta)}{\mathrm{d}\theta}. \tag{4.28a,b}$$

Note that the normal stress $\sigma_{rr} = 0$ on the surface of the sphere is for the incompressible flow, while it is not for a compressible flow. Using Equations (4.21), (4.22) and (4.27), we find that the velocity components $f_2\tilde{\mathbf{u}}_2$ have no contribution to the drag force. Repeating the same procedures, we also find that the other velocity components $f_{0m}^{(-1)}\tilde{\mathbf{u}}_{0m}^{(-1)}$ and $f_{2m}^{(-1)}\tilde{\mathbf{u}}_{2m}^{(-1)}$ have no contribution to the drag force. Consequently, $F_D = F_{D\text{inc}}$, i.e. the compressibility effect does not alter the drag force.

5. Discussion and conclusions

In this paper we have extended the applicability of the Cosserat-spectrum theory to fluid mechanics and have demonstrated its usefulness in the solution of weakly compressible Stokes-flow problems. With the use of a perturbation approach, the solution of a compressible flow (Equations (3.1)) was approximated as the sum of its incompressible counterpart and the corresponding perturbation term. In this perturbation approach, $\varepsilon = M^2/\text{Re} \ll 1$ was assumed for a weakly compressible flow. A general perturbation analysis presented in Section 2 indicates that these terms appear in the order of ε^n, $n = 1, 2, 3 \ldots$. The present paper has retained the first perturbation term.

The continuity equation was employed to evaluate the perturbation terms, which are associated with both the discrete eigenvectors $\tilde{\mathbf{u}}_n$ and the eigenvectors $\tilde{\mathbf{u}}_{nm}^{(-1)}$ corresponding to eigenvalue $\tilde{\omega} = -1$ of infinite multiplicity. Once the Cosserat eigenvalues were obtained and eigenvectors relating to the specific domain Ω derived, the solutions were found by evaluation of the coefficients f_n and $f_{nm}^{(-1)}$ of $\tilde{\mathbf{u}}_n$ and $\tilde{\mathbf{u}}_{nm}^{(-1)}$, respectively. At the present time, the sphere is the only 3D body for which the Cosserat eigenvalues and eigenvectors are available. The velocity-perturbation term is closely related to $\tilde{\mathbf{u}}_n$ and $\tilde{\mathbf{u}}_{nm}^{(-1)}$. The pressure perturbation term has been shown to be related to the divergence of the Cosserat eigenvectors div $\tilde{\mathbf{u}}_n$ and div $\tilde{\mathbf{u}}_{nm}^{(-1)}$, but it could be written in a closed form without involving div $\tilde{\mathbf{u}}_{nm}^{(-1)}$. The density perturbation term was written in a simple closed form without involving both div $\tilde{\mathbf{u}}_n$ and div $\tilde{\mathbf{u}}_{nm}^{(-1)}$.

By applying this solution technique, we have obtained an analytical solution for the compressible Stokes flow over a sphere with a uniform free-stream profile. Only the discrete eigenvector $\tilde{\mathbf{u}}_2$ and finite terms of $\tilde{\mathbf{u}}_{nm}^{(-1)}(n = 0, 2)$ contribute to the velocity field. The velocity, pressure and density perturbation terms were obtained in closed form. Also, the velocity, pressure and density change due to the compressibility effect, and the drag force was shown to remain the same.

Appendix A. The Cosserat eigenvalues and eigenvectors for a spherical rigid inclusion

In the example shown above, we need to use $\tilde{\omega}_n$ (discrete eigenvalue), $\tilde{\mathbf{u}}_n$ (discrete eigenvector), $\tilde{\mathbf{u}}_{0m}^{(-1)}$ and $\tilde{\mathbf{u}}_{2m}^{(-1)}$ (eigenvectors associated with the eigenvalue of infinite multiplicity $\tilde{\omega} = -1$) for the first boundary-value problem of a spherical rigid inclusion in an infinite space. These Cosserat eigenvalues and eigenvectors are recalled as follows [7, pp. 30–32, 189–207].

For an axisymmetric problem, the discrete Cosserat eigenvalue $\tilde{\omega}_n$, the discrete eigenvector $\tilde{\mathbf{u}}_n$ and its divergence are given by

$$\tilde{\omega}_n = -\frac{2n+1}{n+1}, \qquad \tilde{\mathbf{u}}_n = C_n(r^2 - r_0^2)\nabla F_{-(n+1)}, \tag{A1,2}$$

$$\nabla \cdot \tilde{\mathbf{u}}_n = -2(n+1)C_n F_{-(n+1)}, \tag{A3}$$

where $n = 1, 2, \ldots$ and

$$(C_n)^2 = \frac{2n-1}{16(n+1)\pi r_0^3}, \qquad F_{-(n+1)} = \left(\frac{r_0}{r}\right)^{n+1} P_n(\cos\theta), \tag{A4,5}$$

and $P_n(\cos\theta)$ is the Legendre polynomial of degree n.

The Cosserat eigenvector $\tilde{\mathbf{u}}_{0m}^{(-1)}$ and its divergence are given as follows

$$\tilde{\mathbf{u}}_{0m}^{(-1)} = u_{0mr}\mathbf{e}_r + u_{om\theta}\mathbf{e}_\theta, \tag{A6a}$$

$$u_{0mr} = R_{0m}(s), \qquad u_{0m\theta} = 0, \tag{A6b,c}$$

$$R_{0m}(s) = -C_{0m} r_0 s^2 \int_{t=1}^{s} t^{-2} J_{om}(t)\,\mathrm{d}t, \tag{A6d}$$

$$\nabla \cdot \tilde{\mathbf{u}}_{0m}^{(-1)} = C_{0m} s^2 J_{0m}(s), \tag{A7}$$

where $m = 0, 1, 2, \ldots$ and

$$C_{0m}^2 = \frac{1}{4\pi r_0^3 h_{0m}}, \tag{A8}$$

$$J_{0m}(s) = \frac{\Gamma(m+1)}{\Gamma(2m+1)} \sum_{l=0}^{m} (-1)^l \binom{m}{l} \frac{\Gamma(2m-l+1)}{\Gamma(m-l+1)} s^{m-l}, \tag{A9a}$$

$$h_{0m} = \frac{\Gamma^4(m+1)}{(2m+1)\Gamma^2(2m+1)}, \tag{A9b}$$

where $\Gamma(m)$ is the Gamma function and $\binom{m}{l} = m!/l!(m-l)!$ is the binomial coefficient. Equation (A9) defines a Jacobi polynomial with weight $w(s) = 1$, $J_{0m}(s)$, and its norm h_{0m} [15, pp. 773–775].

The first five eigenvectors of $\tilde{\mathbf{u}}_{0m}^{(-1)}$ and their divergence $\nabla \cdot \tilde{\mathbf{u}}_{0m}^{(-1)}(m = 0, 1, \ldots, 4)$ used in the example of uniform flow past a sphere are written out explicitly as follows

$$\tilde{\mathbf{u}}_{00}^{(-1)} = \frac{1}{2\sqrt{\pi r_0}}(s - s^2)\mathbf{e}_r, \tag{A10a}$$

$$\nabla \cdot \tilde{\mathbf{u}}_{00}^{(-1)} = \frac{1}{2\sqrt{\pi r_0^3}} s^2, \tag{A10b}$$

$$\tilde{\mathbf{u}}_{01}^{(-1)} = -\frac{\sqrt{3}}{2\sqrt{\pi r_0}}(3s - 3s^2 + 4s^2\log(s))\mathbf{e}_r, \tag{A10c}$$

$$\nabla \cdot \tilde{\mathbf{u}}_{01}^{(-1)} = -\frac{\sqrt{3}}{2\sqrt{\pi r_0^3}}(3s^2 - 4s^3), \tag{A10d}$$

$$\tilde{\mathbf{u}}_{02}^{(-1)} = \frac{\sqrt{5}}{2\sqrt{\pi r_0}}(6s + 9s^2 - 15s^3 - 20s^2 \log(s))\mathbf{e}_r, \tag{A10e}$$

$$\nabla \cdot \tilde{\mathbf{u}}_{02}^{(-1)} = \frac{\sqrt{5}}{2\sqrt{\pi r_0^3}}(6s^2 - 20s^3 + 15s^4), \tag{A10f}$$

$$\tilde{\mathbf{u}}_{03}^{(-1)} = -\frac{\sqrt{7}}{2\sqrt{\pi r_0}}(10s + 67s^2 - 105s^3 + 28s^4 + 60s^2 \log(s))\mathbf{e}_r, \tag{A10g}$$

$$\nabla \cdot \tilde{\mathbf{u}}_{03}^{(-1)} = -\frac{\sqrt{7}}{2\sqrt{\pi r_0^3}}(10s^2 - 60s^3 + 105s^4 - 56s^5), \tag{A10h}$$

$$\tilde{\mathbf{u}}_{04}^{(-1)} = \frac{3}{2\sqrt{\pi r_0}}(15s + 223s^2 - 420s^3 + 252s^4 - 70s^5 + 140s^2 \log(s))\mathbf{e}_r, \tag{A10i}$$

$$\nabla \cdot \tilde{\mathbf{u}}_{04}^{(-1)} = \frac{3}{2\sqrt{\pi r_0^3}}(15s^2 - 140s^3 + 420s^4 - 504s^5 + 210s^6). \tag{A10j}$$

The Cosserat eigenvector $\tilde{\mathbf{u}}_{2m}^{(-1)}$ and its divergence are as follows

$$\tilde{\mathbf{u}}_{2m}^{(-1)} = u_{2mr}\mathbf{e}_r + u_{2m\theta}\mathbf{e}_\theta, \tag{A11a}$$

$$u_{2mr} = R_{2m}(s)P_2(\cos\theta), \tag{A11b}$$

$$u_{2m\theta} = -s\int_1^s R_{2m}(t)\frac{\mathrm{d}t}{t^2}\frac{\mathrm{d}P_2(\cos\theta)}{\mathrm{d}\theta}, \tag{A11c}$$

$$\begin{aligned} R_{2m}(s) \; &= \; -\tfrac{3}{5}C_{2m}r_0 s^4 \int_{t=1}^{s} (t^{-3} + \tfrac{2}{3}t^2) J_{2m}(t)\,\mathrm{d}t \\ &\quad -\tfrac{2}{5}C_{2m}r_0(s^{-1} - s^4)\int_{t=0}^{s} t^2 J_{2m}(t)\,\mathrm{d}t, \end{aligned} \tag{A11d}$$

$$\nabla \cdot \tilde{\mathbf{u}}_{2m}^{(-1)} = C_{2m}s^3 J_{2m}(s)P_2(\cos\theta). \tag{A12}$$

where $m = 1, 2, 3, \ldots$ and

$$C_{2m}^2 = \frac{5}{4\pi r_0^3 h_{2m}}, \tag{A13}$$

$$J_{2m}(s) = \frac{\Gamma(m+3)}{\Gamma(2m+3)}\sum_{l=0}^{m}(-1)^l\binom{m}{l}\frac{\Gamma(2m-l+3)}{\Gamma(m-l+3)}s^{m-l}, \tag{A14a}$$

$$h_{2m} = \frac{\Gamma^2(m+1)\Gamma^2(m+3)}{(2m+3)\Gamma^2(2m+3)}. \tag{A14b}$$

Equation (A14) defines a Jacobi polynomial with weight $w(s) = s^2$, $J_{2m}(s)$, and its norm h_{2m} [15, pp. 773–775].

The first three eigenvectors of $\tilde{\mathbf{u}}_{2m}^{(-1)}$ and their divergences $\nabla \cdot \tilde{\mathbf{u}}_{2m}^{(-1)} (m = 1, 2, 3)$ used in the example of uniform flow past a sphere are written out explicitly as follows

$$\tilde{\mathbf{u}}_{21}^{(-1)} = -\frac{5}{4\sqrt{\pi r_0}}(s^2 - 4s^3 + 3s^4) P_2(\cos\theta)\mathbf{e}_r + \frac{5}{4\sqrt{\pi r_0}}(s^2 - 2s^3 + s^4)\frac{\mathrm{d}P_2(\cos\theta)}{\mathrm{d}\theta}\mathbf{e}_\theta, \tag{A15a}$$

$$\nabla \cdot \tilde{\mathbf{u}}_{21}^{(-1)} = -\frac{5}{2\sqrt{\pi r_0^3}}(3s^3 - 4s^4) P_2(\cos\theta), \tag{A15b}$$

$$\tilde{\mathbf{u}}_{22}^{(-1)} = \frac{\sqrt{35}}{2\sqrt{\pi r_0}}(s^2 - 10s^3 + 9s^4 - 9s^4\log(s)) P_2(\cos\theta)\mathbf{e}_r - \frac{\sqrt{35}}{2\sqrt{\pi r_0}}(s^2 - 5s^3 + 4s^4 - 3s^4\log(s))\frac{\mathrm{d}P_2(\cos\theta)}{\mathrm{d}\theta}\mathbf{e}_\theta, \tag{A15c}$$

$$\nabla \cdot \tilde{\mathbf{u}}_{22}^{(-1)} = \frac{\sqrt{35}}{2\sqrt{\pi r_0^3}}(6s^3 - 20s^4 + 15s^5) P_2(\cos\theta), \tag{A15d}$$

$$\tilde{\mathbf{u}}_{23}^{(-1)} = -\frac{\sqrt{5}}{2\sqrt{\pi r_0}}(5s^2 - 90s^3 - 27s^4 + 112s^5 - 189s^4\log(s)) P_2(\cos\theta)\mathbf{e}_r + \frac{\sqrt{5}}{2\sqrt{\pi r_0}}(5s^2 - 45s^3 + 12s^4 + 28s^5 - 63s^4\log(s))\frac{\mathrm{d}P_2(\cos\theta)}{\mathrm{d}\theta}\mathbf{e}_\theta, \tag{A15e}$$

$$\nabla \cdot \tilde{\mathbf{u}}_{23}^{(-1)} = -\frac{3\sqrt{5}}{2\sqrt{\pi r_0^3}}(10s^3 - 60s^4 + 105s^5 - 56s^6) P_2(\cos\theta). \tag{A15f}$$

Acknowledgment

The authors would like to thank Professor Markenscoff of UCSD and Professor Paukshto of Saint Petersburg State University, Russia, for their help in the construction of the infinite orthogonal bases $\tilde{\mathbf{u}}^{(-1)}$. Paukshto also pointed out that the pressure perturbation term can be written in a form without involving $\tilde{\mathbf{u}}^{(-1)}$, namely Equation (3.17) in this paper.

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